Unit - III

Solutions of Equations and eigen value problems

part- B.

1. Find the dominant eigenvalue and its eigen vector of the matrix by power method

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}.$$

[May / June 2013]
[Nov / Dec 2014]

Soln:

Let $X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an arbitrary initial eigen vector

$$AX_0 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \\ 0.2 \end{pmatrix} = 5X_1$$

$$AX_1 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 5.2 \\ 0 \\ 2 \end{pmatrix} = 5.2\begin{pmatrix} 1 \\ 0 \\ 0.38 \end{pmatrix} = 5.2\,X_2.$$

$$AX_2 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.38 \end{pmatrix} = \begin{pmatrix} 5.39 \\ 0 \\ 2.95 \end{pmatrix} = 5.39\begin{pmatrix} 1 \\ 0 \\ 0.55 \end{pmatrix} = 5.39\,X_3.$$

$$AX_3 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.55 \end{pmatrix} = \begin{pmatrix} 5.55 \\ 0 \\ 3.75 \end{pmatrix} = 5.55\begin{pmatrix} 1 \\ 0 \\ 0.68 \end{pmatrix} = 5.55\,X_4.$$

$$AX_4 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.68 \end{pmatrix} = \begin{pmatrix} 5.68 \\ 0 \\ 4.4 \end{pmatrix} = 5.68\begin{pmatrix} 1 \\ 0 \\ 0.78 \end{pmatrix} = 5.68\,X_5$$

$$AX_5 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.78 \end{pmatrix} = \begin{pmatrix} 5.78 \\ 0 \\ 4.9 \end{pmatrix} = 5.78\begin{pmatrix} 1 \\ 0 \\ 0.85 \end{pmatrix} = 5.78\,X_6.$$

$$AX_6 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.85 \end{pmatrix} = \begin{pmatrix} 5.85 \\ 0 \\ 5.25 \end{pmatrix} = 5.85\begin{pmatrix} 1 \\ 0 \\ 0.9 \end{pmatrix} = 5.85\,X_7$$

$$AX_7 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.9 \end{pmatrix} = \begin{pmatrix} 5.9 \\ 0 \\ 5.5 \end{pmatrix} = 5.9\begin{pmatrix} 1 \\ 0 \\ 0.93 \end{pmatrix} = 5.8894\,X_8$$

$$AX_8 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.93 \end{pmatrix} = \begin{pmatrix} 5.93 \\ 0 \\ 5.65 \end{pmatrix} = 5.93\begin{pmatrix} 1 \\ 0 \\ 0.95 \end{pmatrix} = 5.93\,X_9.$$

$$A X_9 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.95 \end{pmatrix} = \begin{pmatrix} 5.95 \\ 0 \\ 5.75 \end{pmatrix} = 5.95 \begin{pmatrix} 1 \\ 0 \\ 0.97 \end{pmatrix} = 5.95\, X_{10}$$

$$A X_{10} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.97 \end{pmatrix} = \begin{pmatrix} 5.97 \\ 0 \\ 5.85 \end{pmatrix} = 5.97 \begin{pmatrix} 1 \\ 0 \\ 0.98 \end{pmatrix} = 5.97\, X_{11}$$

$$A X_{11} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.98 \end{pmatrix} = \begin{pmatrix} 5.99 \\ 0 \\ 5.9 \end{pmatrix} = 5.99 \begin{pmatrix} 1 \\ 0 \\ 0.9 \end{pmatrix} = 5.99\, X_{12}$$

$$A X_{12} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.99 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 6\, X_{13}$$

$$A X_{13} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 6\, X_{14}.$$

$\therefore$ dominant eigen value = 6.

corresponding eigen vector is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

2. Find the inverse of the matrix $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 2 \end{pmatrix}$ using Gauss Jordan method. [Nov/Dec 2014]

Soln:

Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 2 \end{pmatrix}$.

$$[A : I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 4 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

Fix R_1 make the first elements of R_2 and R_3 zero using R_1

$$[A : I] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

Fix R_2 make the second elements of R_1 and R_3 zero using R_2.

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$$[A:I] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & -1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 10 & -4 & 3 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 3R_2 \end{array}$$

Fix R_3 make the third elements of R_1 and R_2 zero using R_3

$$[A:I] \approx \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 6 & -2 & 1 \\ 0 & -10 & 0 & 32 & -19 & -3 \\ 0 & 0 & 10 & -4 & 3 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to 5R_1 + R_3 \\ R_2 \to 10R_2 - 3R_3 \end{array}$$

$$[I:A^{-1}] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{6}{5} & \frac{-2}{5} & \frac{1}{5} \\ 0 & 1 & 0 & -32/10 & -19/10 & -3/10 \\ 0 & 0 & 1 & -4/10 & 3/10 & 1/10 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{R_1}{5} \\ R_2 \to \frac{R_2}{-10} \\ R_3 \to \frac{R_3}{10} \end{array}$$

$\therefore$ The inverse of the matrix is

$$A^{-1} = \begin{pmatrix} \frac{6}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{32}{10} & -\frac{19}{10} & -\frac{3}{10} \\ -\frac{4}{10} & \frac{3}{10} & \frac{1}{10} \end{pmatrix}.$$

3. Solve the system of equations using Gauss-elimination method $5x-2y+z=4$, $7x+y-5z=8$ and $3x+7y+4z=10$. [Nov/Dec 2014]

Soln:
The augmented matrix is given by

$$[A,B] = \left[\begin{array}{ccc|c} 5 & -2 & +1 & 4 \\ 7 & 1 & -5 & 8 \\ 3 & 7 & 4 & 10 \end{array}\right].$$

Fix R_1 make the first elements of R_2 and R_3 zero using R_1

$$[A,B] \approx \left[\begin{array}{ccc|c} 5 & -2 & 1 & 4 \\ 0 & 19 & -32 & 12 \\ 0 & 41 & 17 & 38 \end{array}\right] \quad \begin{array}{l} R_2 \to 5R_2 - 7R_1 \\ R_3 \to 5R_3 - 3R_1 \end{array}$$

Fix R_1, R_2 make the second element of R_3 zero using R_2.

$$[A,B] \approx \left[\begin{array}{ccc|c} 5 & -2 & 1 & 4 \\ 0 & 19 & -32 & 12 \\ 0 & 0 & 1635 & 230 \end{array}\right] \quad R_3 \to 19R_3 - 41R_2.$$

using back substitution, $1635z = 230$

$$z = \frac{46}{327}$$

From the second row, we get

$$19y - 32z = 12$$

$$19y - 32\left(\frac{46}{327}\right) = 12.$$

$$19y = 12 + \frac{1472}{327}$$

$$19y = \frac{5396}{327}$$

$$y = \frac{284}{327}$$

From the first row, we get

$$5x - 2y + z = 4.$$

$$5x - \frac{568}{327} + \frac{46}{327} = 4.$$

$$5x = \frac{6408}{327}$$

$$x = \frac{6408}{1635}$$

The solution is $\boxed{x = \frac{6408}{1635},\ y = \frac{284}{327},\ z = \frac{46}{327}.}$

4. compute a real root of $x \log_{10} x - 1.2 = 0$, correct to three decim places by Newton Raphson method. [AU Nov/Dec 2004, May/J 200 May/June 2014].

Soln: Let $f(x) = x \log_{10} x - 1.2$

$$f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e$$

$$= \log_{10} x + \log e = \log_{10} x + 0.4343$$

$f(1) = \log_{10} 1 - 1.2 = -1.2 = -ve$

$f(1.5) = 1.5 \log_{10} 1.5 - 1.2 = -0.9359 = -ve.$

$f(2) = 2 \log_{10} 2 - 1.2 = -0.5979 = -ve.$

$f(3) = 3 \log_{10} 3 - 1.2 = 0.2314 = +ve$

Since $f(2)$ and $f(3)$ are opposite in sign, a root lies betw 2 and 3.

Here $|f(2)| > |f(3)|$.

$\therefore$ The root is nearer to 3.

Let $x_0 = 2.7$.

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By Newton-Raphson formula,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.7 - \frac{f(2.7)}{f'(2.7)}$$

$$= 2.7 - \left[\frac{2.7 \log_{10} 2.7 - 1.2}{0.4343 + \log_{10} 2.7}\right]$$

$$= 2.7 - \left(\frac{-0.035}{0.867}\right)$$

$$= 2.7 + \frac{0.035}{0.867}$$

$$= 2.740$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.740 - \frac{f(2.74)}{f'(2.74)}$$

$$= 2.74 - \left[\frac{2.74 \log_{10}(2.74) - 1.2}{0.4343 + \log_{10} 2.74}\right]$$

$$= 2.74 - \left(\frac{-0.0006}{0.872}\right)$$

$$= 2.74 + \frac{0.0006}{0.872}$$

$$= 2.741.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.741 - \frac{f(2.741)}{f'(2.741)}$$

$$= 2.741 - \left[\frac{(2.741) \log_{10} 2.741 - 1.2}{0.4343 + \log_{10} 2.741}\right]$$

$$= 2.741 - \frac{0.003}{0.872}$$

$$= 2.741.$$

Here $x_2 = x_3$.

$\therefore$ The better approximation is 2.741.

5. Using Gauss Jordan method, find the inverse of

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$$

[AU, NOV 2004, April 2010, Oct '96]
[AU M/J 2012]

Soln:

$$\text{Let } [A, I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right] \quad R_2 \to \frac{R_2}{2}.$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right] \quad R_3 \to \frac{R_3}{-4}.$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 6R_3 \\ R_2 \to R_2 + 3R_3. \end{array}$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{pmatrix}.$$

$$= \frac{1}{4}\begin{pmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{pmatrix}$$

6. Solve the following system of equations using Gauss Seidal Method. (63)

$$27x + 6y - z = 85$$
$$x + y + 54z = 110 \qquad [\text{May/June } 2012]$$
$$6x + 15y + 2z = 72$$

Soln:- As the co-efficient matrix is not diagonally dominant, we rewrite the equations

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Since the diagonal elements are dominant in the co-efficient matrix, we write x, y, z as follows:

$$x = \frac{1}{27}[85 - 6y + z]$$
$$y = \frac{1}{15}[72 - 6x - 2z]$$
$$z = \frac{1}{54}[110 - x - y]$$

Let the initial values be $y = 0,\ z = 0$

First iteration:

$$x^{(1)} = \frac{1}{27}[85 - 6y^{(0)} + z^{(0)}] = \frac{1}{27}[85 - 6(0) + 0] = 3.148$$
$$y^{(1)} = \frac{1}{15}[72 - 6x^{(1)} - 2z^{(0)}] = \frac{1}{15}[72 - 6(3.148) - 0] = 3.541$$
$$z^{(1)} = \frac{1}{54}[110 - x^{(1)} - y^{(1)}] = \frac{1}{54}[110 - 3.148 - 3.541] = 1.913.$$

Second iteration:

$$x^{(2)} = \frac{1}{27}[85 - 6y^{(1)} + z^{(1)}] = \frac{1}{27}[85 - 6(3.541) + 1.913] = 2.432.$$
$$y^{(2)} = \frac{1}{15}[72 - 6x^{(2)} - 2z^{(1)}] = \frac{1}{15}[72 - 6(2.432) - 2(1.913)] = 3.572$$
$$z^{(2)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}] = \frac{1}{54}[110 - 2.432 - 3.572] = 1.926.$$

Third iteration:

$$x^{(3)} = \frac{1}{27}[85 - 6y^{(2)} + z^{(2)}] = \frac{1}{27}[85 - 6(3.572) + 1.926] = 2.426$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6x^{(3)} - 2z^{(2)}\right] = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.57$$

$$z^{(3)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right] = \frac{1}{54}\left[110 - 2.426 - 3.573\right] = 1.926$$

Fourth iteration:

$$x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} + z^{(3)}\right] = \frac{1}{27}\left[85 - 6(3.573) + 1.926\right] = 2.426$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6x^{(4)} - 2z^{(3)}\right] = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.573.$$

$$z^{(4)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right] = \frac{1}{54}\left[110 - 2.426 - 3.573\right] = 1.926.$$

Hence $x = 2.426$, $y = 3.573$, $z = 1.926$.

7. Solve by Gauss-Seidal iteration method the system $5x - y + z = 10$; $2x + 4y = 12$; $x + y + 5z = -1$ [AU Apr/May '10, '13]

Soln:

The diagonal elements are dominant in the co-efficient matrix.

We write x, y and z as follows.

$$x = \frac{1}{5}\left[10 + y - z\right]$$

$$y = \frac{1}{4}\left[12 - 2x\right]$$

$$z = \frac{1}{5}\left[-1 - x - y\right]$$

Let the initial values be $y = 0$, $z = 0$.

First iteration:

$$x^{(1)} = \frac{1}{5}\left[10 + y^{(0)} - z^{(0)}\right] = \frac{1}{5}\left[10\right] = 2$$

$$y^{(1)} = \frac{1}{4}\left[12 - 2x^{(1)}\right] = \frac{1}{4}\left[12 - 2(2)\right] = 2.$$

$$z^{(1)} = \frac{1}{5}\left[-1 - x^{(1)} - y^{(1)}\right] = \frac{1}{5}\left[-1 - 2 - 2\right] = -1.$$

Second iteration:

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$x^{(2)} = \frac{1}{5}[10 + y^{(1)} - z^{(1)}] = \frac{1}{5}[10+2+1] = 2.6$

$y^{(2)} = \frac{1}{4}[12 - 2x^{(2)}] = \frac{1}{4}[12 - 2(2.6)] = 1.7$

$z^{(2)} = \frac{1}{5}[-1 - x^{(2)} - y^{(2)}] = \frac{1}{5}[-1 - 2.6 - 1.7] = -1.06.$

Third iteration:

$x^{(3)} = \frac{1}{5}[10 + y^{(2)} - z^{(2)}] = \frac{1}{5}[10 + 1.7 + 1.06] = 2.552$

$y^{(3)} = \frac{1}{4}[12 - 2x^{(3)}] = \frac{1}{4}[12 - 2(2.552)] = 1.724.$

$z^{(3)} = \frac{1}{5}[-1 - x^{(3)} - y^{(3)}] = \frac{1}{5}[-1 - 2.552 - 1.724] = -1.055.$

Fourth iteration:

$x^{(4)} = \frac{1}{5}[10 + y^{(3)} - z^{(3)}] = \frac{1}{5}[10 + 1.724 + 1.055] = 2.556$

$y^{(4)} = \frac{1}{4}[12 - 2x^{(4)}] = \frac{1}{4}[12 - 2(2.556)] = 1.722$

$z^{(4)} = \frac{1}{5}[-1 - x^{(4)} - y^{(4)}] = \frac{1}{5}[-1 - 2.556 - 1.722] = -1.056.$

Fifth iteration:

$x^{(5)} = \frac{1}{5}[10 + y^{(4)} - z^{(4)}] = \frac{1}{5}[10 + 1.722 + 1.056] = 2.556$

$y^{(5)} = \frac{1}{4}[12 - 2x^{(5)}] = \frac{1}{4}[12 - 2(2.556)] = 1.722$

$z^{(5)} = \frac{1}{5}[-1 - x^{(5)} - y^{(5)}] = \frac{1}{5}[-1 - 2.556 - 1.722] = -1.056.$

Hence $x = 2.556$, $y = 1.722$, $z = -1.056$.

8. Solve by Gauss Elimination Method

$x_1 + x_2 + x_3 + x_4 = 2$

$2x_1 - x_2 + 2x_3 - x_4 = -5$

$3x_1 + 2x_2 + 3x_3 + 4x_4 = 7$ [Nov/Dec 2013]

$x_1 - 2x_2 - 3x_3 + 2x_4 = 5$

Soln:

The augmented matrix is given by

$$[A, B] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 2 & -1 & -5 \\ 3 & 2 & 3 & 4 & 7 \\ 1 & -2 & -3 & 2 & 5 \end{array}\right]$$

Fix R_1 make the first elements of R_2, R_3 and R_4 zero using R_1

$$[A, B] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

Fix R_1 and R_2, make the second elements of R_3 and R_4 zero using R_2.

$$[A, B] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & 0 & 0 & 6 & 12 \\ 0 & 0 & -4 & 4 & 12 \end{array}\right] \quad \begin{array}{l} R_3 \to 3R_3 - R_2 \\ R_4 \to R_4 - R_2. \end{array}$$

Interchange R_3 and R_4,

$$[A, B] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & 0 & -4 & 4 & 12 \\ 0 & 0 & 0 & 6 & 12 \end{array}\right]$$

using back substitution, we get

$$6x_4 = 12$$
$$\boxed{x_4 = 2}.$$

From the 3rd row,
$$-4x_3 + 4x_4 = 12.$$
$$-4x_3 + 4(2) = 12$$
$$-4x_3 = 12 - 8$$
$$-4x_3 = 4$$
$$\boxed{x_3 = -1}$$

From the 2nd row,
$$-3x_2 - 3x_4 = -9$$
$$-3x_2 - 3(2) = -9.$$
$$-3x_2 = -9 + 6$$
$$-3x_2 = -3$$
$$\boxed{x_2 = 1}$$

From the first row,
$$x_1 + x_2 + x_3 + x_4 = 2.$$
$$x_1 + 1 - 1 + 2 = 2.$$
$$\boxed{x_1 = 0}$$

The solution is $x_1 = 0$, $x_2 = 1$, $x_3 = -1$, $x_4 = 2$.

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9. Find the dominant eigen value and the corresponding Eigen vector of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Find also the least latent root and hence find the third eigen value.

[AU M/J 2007, 2008, 2010]

(or)

Find all the eigen values of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Soln:

Let $X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an arbitrary initial eigen vector

$$AX_0 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1X_1$$

$$AX_1 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = 7X_2$$

$$AX_2 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.5714 \\ 1.8572 \\ 0 \end{pmatrix} = 3.5714\begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix} = 3.5714X_3$$

$$AX_3 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix} = \begin{pmatrix} 4.12 \\ 2.04 \\ 0 \end{pmatrix} = 4.12\begin{pmatrix} 1 \\ 0.4951 \\ 0 \end{pmatrix} 4.12(X_4)$$

$$AX_4 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4951 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.9706 \\ 1.9902 \\ 0 \end{pmatrix} = 3.9706\begin{pmatrix} 1 \\ 0.5012 \\ 0 \end{pmatrix} = 3.9706X_5$$

$$AX_5 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.5012 \\ 0 \end{pmatrix} = \begin{pmatrix} 4.0072 \\ 2.0024 \\ 0 \end{pmatrix} = 4.0072\begin{pmatrix} 1 \\ 0.4997 \\ 0 \end{pmatrix} = 4.0072X_6$$

$$AX_6 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4997 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.9982 \\ 1.9994 \\ 0 \end{pmatrix} = 3.9982\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 3.9982X_7$$

$$AX_7 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 4X_8$$

$$AX_8 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 4X_9.$$

$\therefore$ Dominant eigen value $= 4$.

corresponding eigen vector $= \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$.

To find least eigen value.

Let $B = A - 4I$. Since $\lambda = 4$.

$$B = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix}.$$

Now, find the dominant eigen value of B.

Let $Y_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an arbitrary initial eigen vector.

$$BY_0 = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = -3\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = -3Y_1$$

$$BY_1 = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 1.6666 \\ 0 \end{pmatrix} = -5\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = -5Y_2$$

$$BY_2 = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 1.6666 \\ 0 \end{pmatrix} = -5\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix}$$

$\therefore$ Dominant eigen value of $B = -5$.

$\therefore$ Smallest eigen value of $A = -5+4 = -1$.

Sum of the eigen values = Sum of the diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 1+2+3 = 6$$

ie) $4 + (-1) + \lambda_3 = 6$

$$\lambda_3 = 3.$$

$\therefore$ All the three eigen values are $\{ 4, 3, -1.$

10. Solve by Jacobi's iteration method the system for $8x-3y+2z=20$, $6x+3y+12z=35$; $4x+11y-z=33$. [Nov/Dec '2010]

Soln: Given

$$8x-3y+2z=20$$
$$6x+3y+12z=35$$
$$4x+11y-z=33.$$

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Since the diagonal elements are not dominant in the iteration method, ~~we~~ interchanging the equations we get

$$8x-3y+2z=20$$
$$4x+11y-z=33$$
$$6x+3y+12z=35$$

$$\Rightarrow \quad x=\frac{1}{8}[20+3y-2z]$$
$$y=\frac{1}{11}[33-4x+z]$$
$$z=\frac{1}{12}[35-6x-3y]$$

Let the initial approximation be $x_0=0,\ y_0=0,\ z_0=0$.

First iteration:

$$x^{(1)}=\frac{1}{8}[20+3y_0-2z_0]$$
$$=\frac{1}{8}[20+3(0)-2(0)]=2.5$$
$$y^{(1)}=\frac{1}{11}(33-4x_0+z_0)=\frac{1}{11}(33-4(0)+0)=3.$$
$$z^{(1)}=\frac{1}{12}[35-6x_0-3y_0]=\frac{1}{12}[35-6(0)-3(0)]=2.9167.$$

Second iteration:

$$x^{(2)}=\frac{1}{8}(20+3y^{(1)}-2z^{(1)})=\frac{1}{8}[20+3(3)-(2.9167)2]=2.89!$$
$$=2.3561.$$
$$y^{(2)}=\frac{1}{11}[33-4x^{(1)}+z^{(1)}]=\frac{1}{11}[33-4(2.5)+2.9167]=0.9167.$$
$$z^{(2)}=\frac{1}{12}[35-6x^{(1)}-3y^{(1)}]=\frac{1}{12}[35-6(2.5)-3(3)]$$

Third iteration:

$$x^{(3)}=\frac{1}{8}[20+3y^{(2)}-2z^{(2)}]=\frac{1}{8}[20+3(2.32561)-2(0.9167)]$$
$$=3.1544$$
$$y^{(3)}=\frac{1}{11}[33-4x^{(2)}+z^{(2)}]=\frac{1}{11}[33-4(2.8958)+0.9167]$$
$$=2.0303.$$
$$z^{(3)}=\frac{1}{12}[35-6(2.8958)-3(2.3561)]=0.8797.$$

Fourth iteration:

$x^{(4)} = \frac{1}{8}[20 + 3y^{(3)} - 2z^{(3)}] = \frac{1}{8}[20 + 3(2.0303) - 2(0.8797)] = 3.0414$

$y^{(4)} = \frac{1}{11}[33 - 4x^{(3)} + z^{(3)}] = \frac{1}{11}[33 - 4(3.1544) + 0.8797] = 1.9329.$

$z^{(4)} = \frac{1}{12}[35 - 6x^{(3)} - 3y^{(3)}] = \frac{1}{12}[35 - 6(3.1544) - 3(2.0303)] = 0.8319.$

Fifth iteration:

$x^{(5)} = \frac{1}{8}[20 + 3y^{(4)} - 2z^{(4)}] = \frac{1}{8}[20 + 3(1.9329) - 2(0.8319)] = 3.0169$

$y^{(5)} = \frac{1}{11}[33 - 4x^{(4)} + z^{(4)}] = \frac{1}{11}[33 - 4(3.0414) + 0.8319] = 1.9697.$

$z^{(5)} = \frac{1}{12}[35 - 6x^{(4)} - 3y^{(4)}] = \frac{1}{12}[35 - 6(3.0414) - 3(1.9329)] = 0.9127$

Sixth iteration:

$x^{(6)} = \frac{1}{8}[20 + 3y^{(5)} - 2z^{(5)}] = \frac{1}{8}[20 + 3(1.9697) - 2(0.9127)] = 3.010$

$y^{(6)} = \frac{1}{11}[33 - 4x^{(5)} + z^{(5)}] = \frac{1}{11}[33 - 4(3.0169) + 0.9127] = 1.9859$

$z^{(6)} = \frac{1}{12}[35 - 6x^{(5)} - 3y^{(5)}] = \frac{1}{12}[35 - 6(3.0169) - 3(1.9697)] = 0.9158$

Seventh iteration:

$x^{(7)} = \frac{1}{8}[20 + 3y^{(6)} - 2z^{(6)}] = \frac{1}{8}[20 + 3(1.9859) - 2(0.9158)]$

$= 3.0158$

$y^{(7)} = \frac{1}{11}[33 - 4x^{(6)} - z^{(6)}]$

$= \frac{1}{11}[33 - 4(3.0105) + 0.9158] = 1.9885$

$z^{(7)} = \frac{1}{12}[35 - 6x^{(6)} - 3y^{(6)}]$

$= \frac{1}{12}[35 - 6(3.0105) - 3(1.9859)] = 0.9149.$

Eighth iteration:

$x^{(8)} = \frac{1}{8}[20 + 3y^{(7)} - 2z^{(7)}]$

$= \frac{1}{8}[20 + 3(1.9885) - 2(0.9149)]$

$= 3.0170$

$y^{(8)} = \frac{1}{11}[33 - 4(3.0158) + 0.9149] = 1.9865$

$z^{(8)} = \frac{1}{12}\left[35 - 6x^{(7)} - 3y^{(7)}\right]$ (91)

$= \frac{1}{12}\left[35 - 6(3.0158) - 3(1.9885)\right] = 0.9116.$

Ninth iteration

$x^{(9)} = \frac{1}{8}\left[20 + 3y^{(8)} - 2z^{(8)}\right] = \frac{1}{8}\left[20 + 3(1.9865) - 2(0.9116)\right]$

$= 3.0170$

$y^{(9)} = \frac{1}{11}\left[33 - 4x^{(8)} + z^{(8)}\right]$

$= \frac{1}{11}\left[33 - 4(3.0170) + 0.9116\right]$

$= 1.9858$

$z^{(9)} = \frac{1}{12}\left[35 - 6x^{(8)} - 3y^{(8)}\right]$

$= \frac{1}{12}\left[35 - 6(3.0170) - 3(1.9865)\right]$

$= 0.9115$

Tenth iteration

$x^{(10)} = \frac{1}{8}\left[20 + 3y^{(9)} - 2z^{(9)}\right] = \frac{1}{8}\left[20 + 3(1.9858) - 2(0.9115)\right] = 3.0168$

$y^{(10)} = \frac{1}{11}\left[33 - 4x^{(9)} + z^{(9)}\right] = \frac{1}{11}\left[33 - 4(3.0170) + 0.9115\right] = 1.9858$

$z^{(10)} = \frac{1}{12}\left[35 - 6x^{(9)} - 3y^{(9)}\right] = \frac{1}{12}\left[35 - 6(3.0170) - 3(1.9858)\right] = 0.9117$

Eleventh iteration

$x^{(11)} = \frac{1}{8}\left[20 + 3y^{(10)} - 2z^{(10)}\right] = \frac{1}{8}\left[20 + 3(1.9858) - 2(0.9117)\right] = 3.0168$

$y^{(11)} = \frac{1}{11}\left[33 - 4x^{(10)} + z^{(10)}\right] = \frac{1}{11}\left[33 - 4(3.0168) + 0.9117\right] = 1.9859$

$z^{(11)} = \frac{1}{12}\left[35 - 6x^{(10)} - 3y^{(10)}\right] = \frac{1}{12}\left[35 - 6(3.0168) - 3(1.9858)\right]$

$= 0.9118$

Twelfth iteration

$x^{(12)} = \frac{1}{8}\left[20 + 3y^{(11)} - 2z^{(11)}\right] = \frac{1}{8}\left[20 + 3(1.9858) - 2(0.9118)\right]$

$= 3.0168$

$y^{(12)} = \frac{1}{11}\left[33 - 4x^{(11)} + z^{(11)}\right]$

$$= \frac{1}{11}\left[33 - 4(3.0168) + 0.9118\right] = 1.9859.$$

$$z^{(12)} = \frac{1}{12}\left[35 - 6x^{(11)} - 3y^{(11)}\right]$$

$$= \frac{1}{12}\left[35 - 6(3.0168) - 3(1.9858)\right] = 0.9118$$

From 11^{th} and 12^{th} iterations,

$x = 3.0168$, $y = 1.9858$, $z = 0.9118$ (correct to four decimal places)

Excercise

1. Using Newton Raphson method, establish the formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ to calculate the square root of N. And find the square root of 5 correct to four places of decimals

2. Solve $10x + y + z = 12$, $2x + 10y + z = 13$, $x + y + 5z = 7$ by (i) Gauss elimination method (ii) Gauss Jordan method

3. Solve by Gauss - Seidel method, $2x + y + 6z = 9$, $8x + 3y + 2z = 13$, $x + 5y + z = 7$

4. Find the inverse of the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{pmatrix}$ by Gauss Jordan method

5. Using power method, find the largest eigen values and the corresponding eigen vector of the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$