UNIT - III

NONLINEAR AND RANDOM VIBRATION

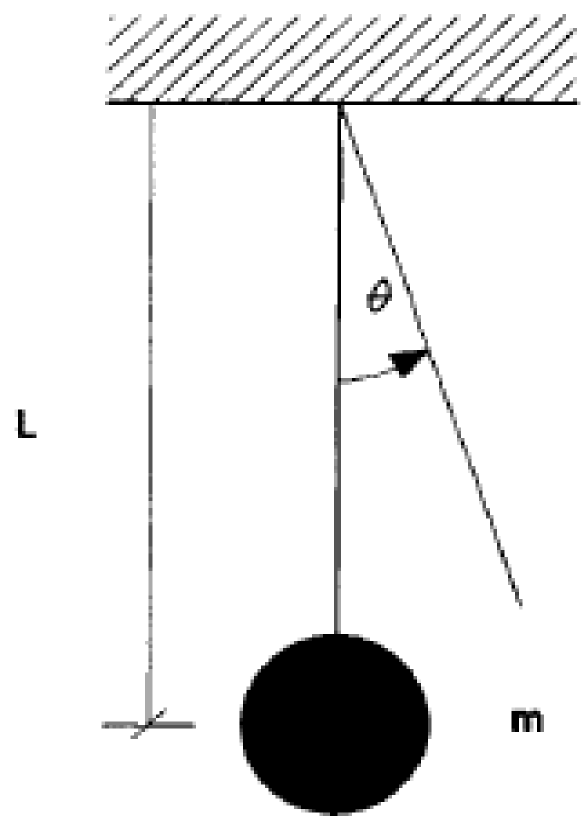
Introduction to Nonlinear Vibrations:

The progress achieved in the past decades in the applied mechanics field is attributed to the representation of complex physical problems by simple mathematical equations. In many applications, these equations are nonlinear. In spite of this fact, simplifications consistent with the physical situation permit, in most cases, a linearization process that simplifies the mathematical solution of the problem while conserving the precision of the physical results. However, in few cases, the linear solutions are not sufficient to describe adequately the problem at hand because new physical phenomena are introduced and can be explained only if nonlinearity is considered.

Simple Examples of Nonlinear Systems:

- Simple Pendulum in Free Vibrations:-

Consider the simple pendulum in free vibrations shown in Figure.



Simple pendulum in free vibrations

The equation of motion of the pendulum can be written as

$$mL^2\theta'' + mgL\sin\theta = 0$$

Can be written as

$$mL^2\theta'' + mgL\,\theta - mgL\frac{\theta^3}{3!} + mgL\frac{\theta^5}{5!} - \cdots = 0$$

.

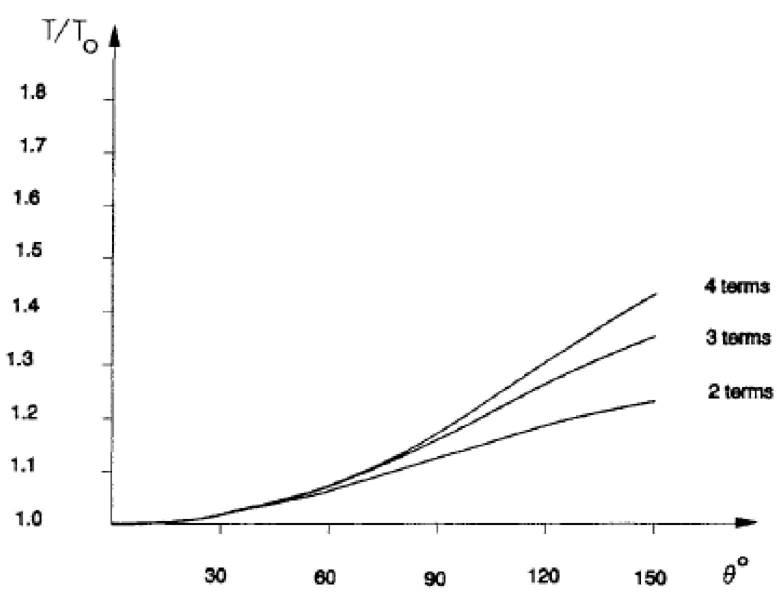
Physical Properties of Nonlinear Systems:

- Undamped Free Vibrations:

Physical considerations reveal that, for a mechanical system with nonlinear stiffness in free vibrations, the period (and thus the frequency) of the response will be a function of the amplitude of vibration. This is expected mathematically since k = k(x) and therefore T = T(x). It is to be emphasized that the natural frequency is a constant and is a property of the mechanical system, despite whether the system is linear. The frequency of response in free vibration of a linear system is constant and is equal to the natural frequency of the system, while a nonlinear system in free vibration responds with a frequency that is a function of the amplitude of vibration. As an example (the proof will be given in the next sections), for the dependence of the period of free vibration on the amplitude of the response, it can be shown that the period of the simple pendulum of Fig. is given by

$$T = T_0\left[1 + \frac{1}{4}\left(\sin\frac{\theta}{2}\right)^2 + \frac{9}{64}\left(\sin\frac{\theta}{2}\right)^4 + \frac{25}{256}\left(\sin\frac{\theta}{2}\right)^6 + \cdots\right]$$

Where TO is the period of the linear system. A plot of T / To vs 0 is shown in Fig.



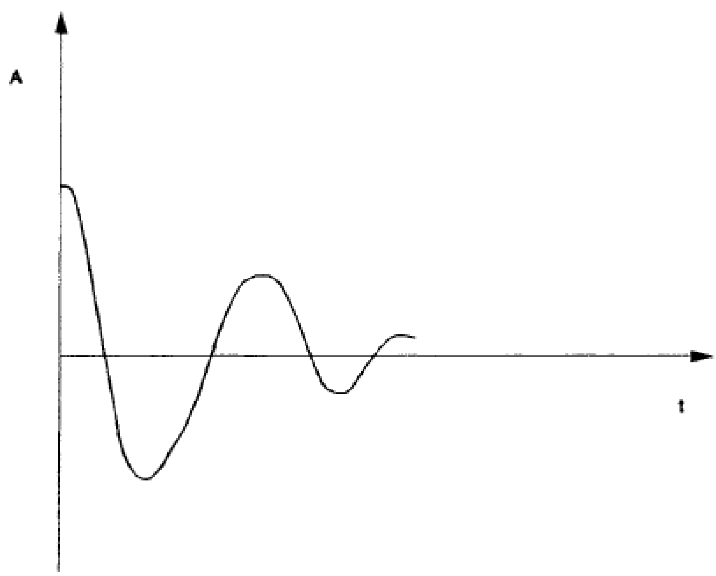
Period of free vibrations of a simple pendulum

- Damped Free Vibrations

Consider a nonlinear damped system having a hard spring nonlinearity characteristic in free vibrations. The system equation of motion can be written as

$$mx'' + cx' + k_0x + k_1x^3 = 0$$

With initial conditions different from zero and an initial displacement value in the nonlinear regime, physical considerations and Eq. (5.10) reveal that the response will appear as the curve sketched in Fig. We notice that, for nonlinear amplitude values, we will have smaller periods of response (thus higher frequencies) compared to the linear part. Thus, we expect that the amplitude of the response will begin with a certain value in the nonlinear regime, and the system will oscillate with frequencies higher than the damped natural frequency; with the increase of time, the amplitude of the response will decrease due to the system damping. As a result, we will have an amplitude response oscillating with a decrease in amplitude and frequency values until it reaches the linear amplitude where the system responds with damped amplitude and a constant frequency equal to the system damped natural frequency.



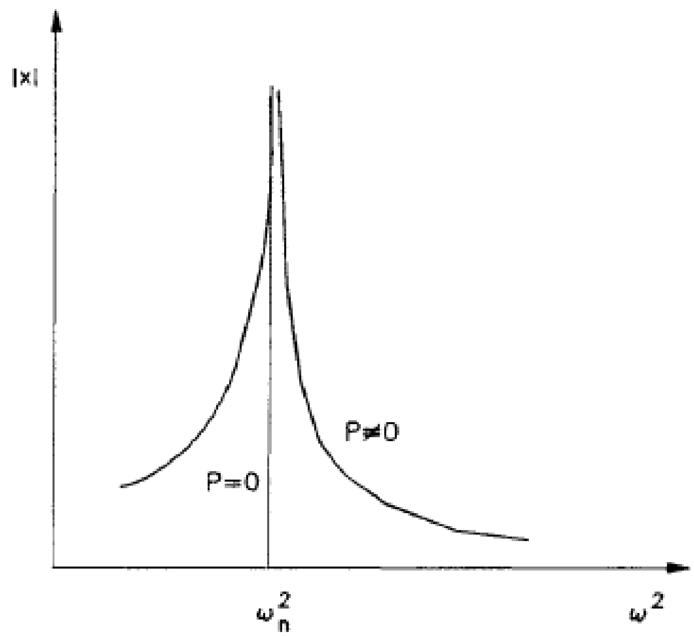
Damped free vibration response of a nonlinear system

- Forced Vibrations

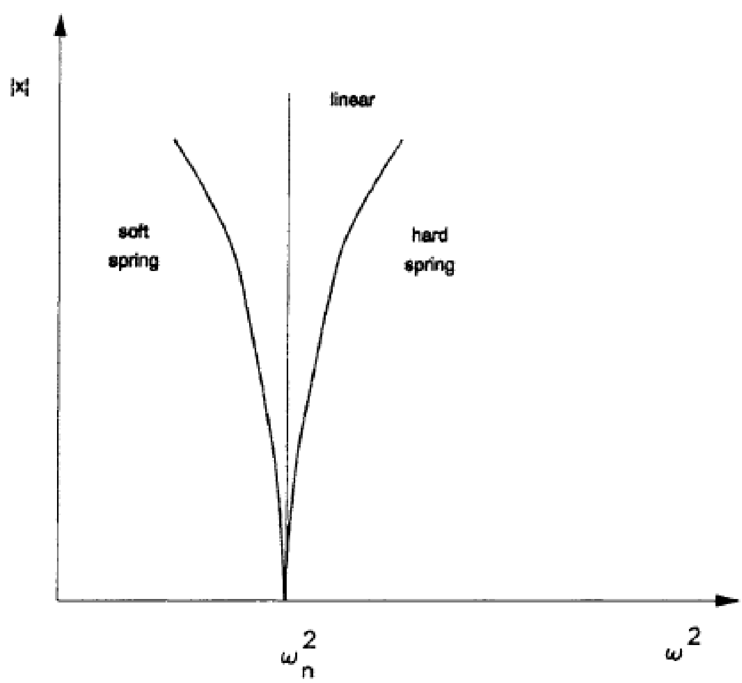
Consider an undamped linear single degree of freedom with a harmonic external excitation. The equation of motion of the system reads

$$x'' + \omega_n^2 x = \frac{P}{m} \cos \omega t$$

The amplitude of the permanent response is sketched in Fig. We notice that for P = 0, i.e., for free vibration, we will have a harmonic response with a frequency of response equal to the undamped natural frequency of the system.



Permanent response amplitude of a linear undamped system due to harmonic external excitation. We expect that the amplitude of the response when plotted against the frequency of excitation will have the form sketched in Fig. for soft and hard springs, respectively.



Free vibration response of linear and nonlinear systems

Solutions of the Equation of Motion of a Single-Degree-of-Freedom Nonlinear System:

- Exact Solutions:

Very few nonlinear differential equations have exact solutions. Exact mathematical solutions of nonlinear systems are studied not only because of their importance for the cases where they exist but also because these exact solutions can be used in the studies of the performance and convergence of nonlinear numerical algorithm solvers that are to be used for the solution of the problems that do not have exact solutions.

- Free vibration:

Consider an undamped single-degree-of- freedom system with stiffness nonlinearity in free vibration. The related equation of motion can be written as

$$x'' + \phi^2 f(x) = 0$$

Can be written as

$$\frac{d(x')^2}{dx} + 2\phi^2 f(x) = 0$$

Integrating, we obtain

$$(x')^2 = 2\phi^2 \int_x^X f(\xi)\, d\xi$$

We now consider the case when f (x) is given by

$$f(x) = x^n + \mu x^m \qquad m > n > 0$$

$$m = 3, 5, 7, \ldots \qquad n = 1, 3, 5, \ldots$$

We obtain

$$T = \frac{4}{\phi\sqrt{X^{n-1}}}\left[\sqrt{\frac{n+1}{2}}\int_0^1 \frac{du}{\sqrt{(1+\nu) - (u^{n+1} + \nu u^{m+1})}}\right]$$

Where

$$\nu = \mu X^{m-n}\left\{\frac{n+1}{m+1}\right\}$$

The extension to the case of a higher-order polynomial is straightforward.

- Forced vibration:

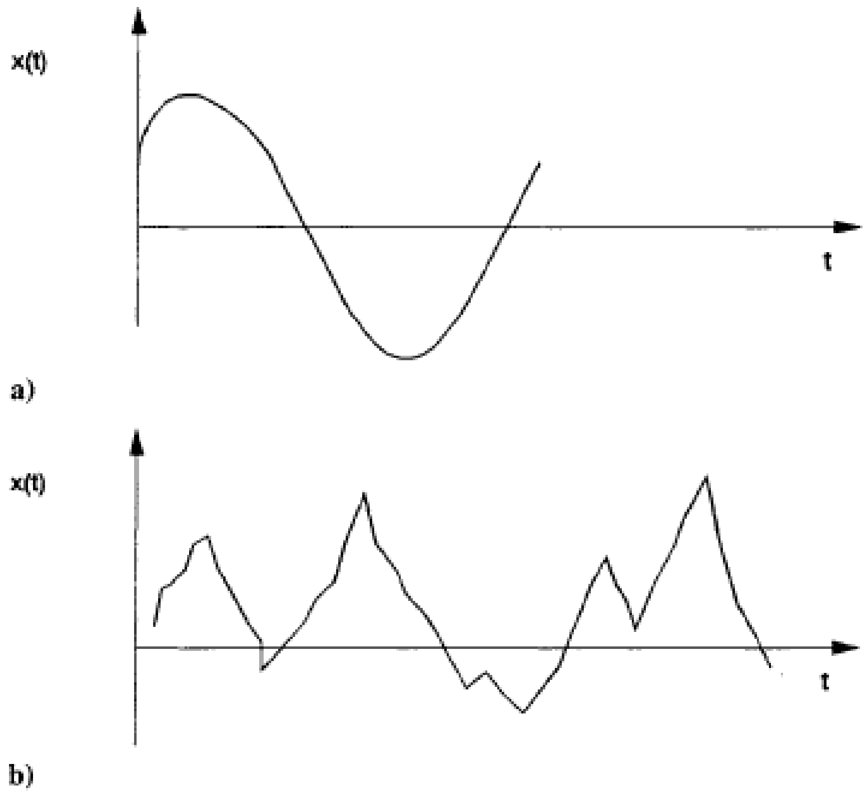
There is no exact solution for the general case of forced vibration of a nonlinear dynamic single-degree-of-freedom system. The solutions are therefore obtained using numerical methods that will be discussed in the next section.

Multi degree-of-Freedom Nonlinear Systems:

The step-by-step numerical integration methods given in previous chapter are directly extended for the analysis of arbitrary nonlinear systems with multiple degrees of freedom. As in the linear case, the time-history response is divided into short, normally equal time increments, and the response is calculated at the end of the time interval for a linearized system having properties determined at the beginning of the interval. The system nonlinear properties are then modified at the end of the interval to conform to the state of deformations and stresses at that time. The mass matrix is usually constant in most practical applications so that its inverse is evaluated once at the beginning of the solution procedure. The stiffness and the damping matrices are modified at the beginning of each step. Therefore, during each step of the nonlinear solution, a triangular decomposition of the equivalent stiffness matrix must be done to obtain the end displacements and velocities. As in the linear case, the acceleration vectors are obtained solving the equations of motion at the beginning of the interval to avoid accumulation of errors during the solution procedure. The modal transformation technique can be used in the solution of the nonlinear system with multiple degrees of freedom; however, in this case, the related matrices are coupled, but the system will have a smaller number of equations compared to the original system written in the physical coordinates. The step-by-step integration procedures are applied to the transformed smaller system of equations.

Introduction to random vibrations:

Consider the record of a measured variable x (t), illustrated in Fig.1, which can represent for instance the displacement of a point in a structure as a function of time. In Fig.la, we can conclude that the variable x(t) is predominantly harmonic, while x(t) of Fig.1b is predominantly irregular. If we repeat the process of measuring and recording the response of the displacement several times and if in all cases we obtain the same responses in both processes, we define such processes as being deterministic processes. Now if, in the process of Fig.la, during the repeated measurements of the records at each time, we obtain a different angle of phase and if, in the process of Fig.1b, the responses are different from each other during the repeated measurements, we call such processes random processes. Random processes are characterized by the fact that their behavior cannot be predicted in advance and therefore can be treated only in a statistical manner. We will begin this chapter by studying random processes and their statistical properties. In the sequence, we will study the response of linear systems due to random excitations.

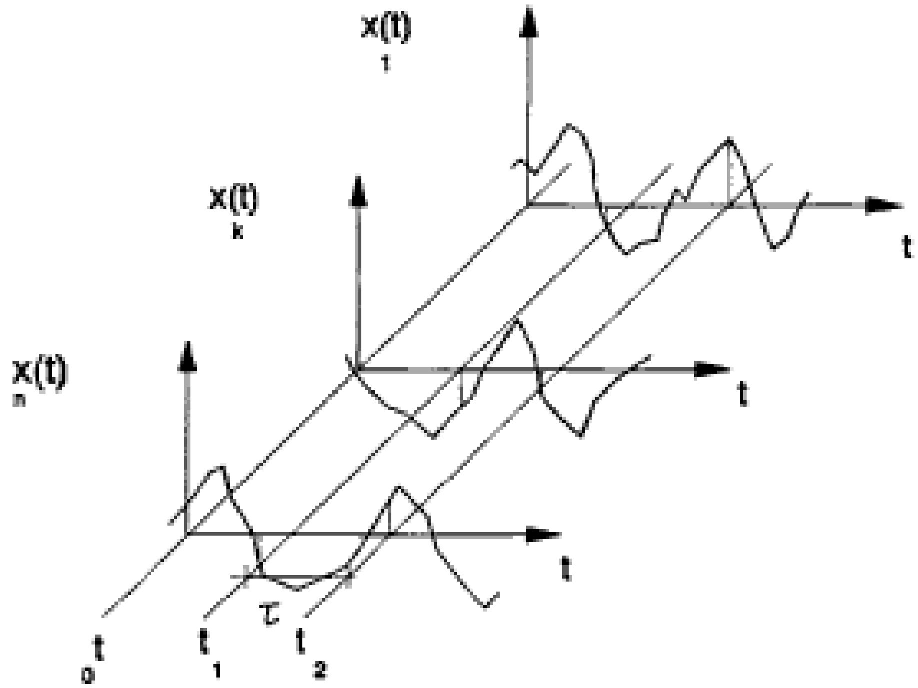


1 Record of a variable as a function of time

Classification of Random Processes:

- Stationary Random Processes:

Consider n records of a random variable as given in Fig.2. We define the complete set of xk(t),k = 1 , 2 , . . . , n as a random process, and each record of the set will be called a sample of the random process. Consider now the values of xk(t) for the instant of time t = t\; we can write the mean value of the random process at that instant of time as



2 Time history of a random process

$$\mu_x(t_1) = \frac{1}{n}\sum_{k=1}^{n} x_k(t_1)$$

For an instant of time t = t^ separated from t\ by an interval of time r, we can write a statistical measurement of the behavior of the mean value in relation to a shift r as a function Rx(t\, t\ + r), given by

$$R_x(t_1, t_1 + \tau) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k(t_1)x_k(t_1 + \tau)$$

For example, for two shifts, we can write an expression in the form

$$R_x(t_1, t_1 + \tau, t_1 + \sigma) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k(t_1)x_k(t_1 + \tau)x_k(t_1 + \sigma)$$

In general, *iix(t\\ Rx(t* , *t* + T), *Rx(t\ ,ti+T,t*+ cr), etc., will be functions of *t* where the mean values have been calculated. Now if in a random process these mean values do not depend on *t*, i.e., ^*x(t\) = JJLX* = const and *Rx(t\, t\ + r)* = *Rx(r)* and *Rx(t\, t* + r, *t\ + a)* = *Rx(r,* a), etc., we call the random process a process that is heavily stationary.

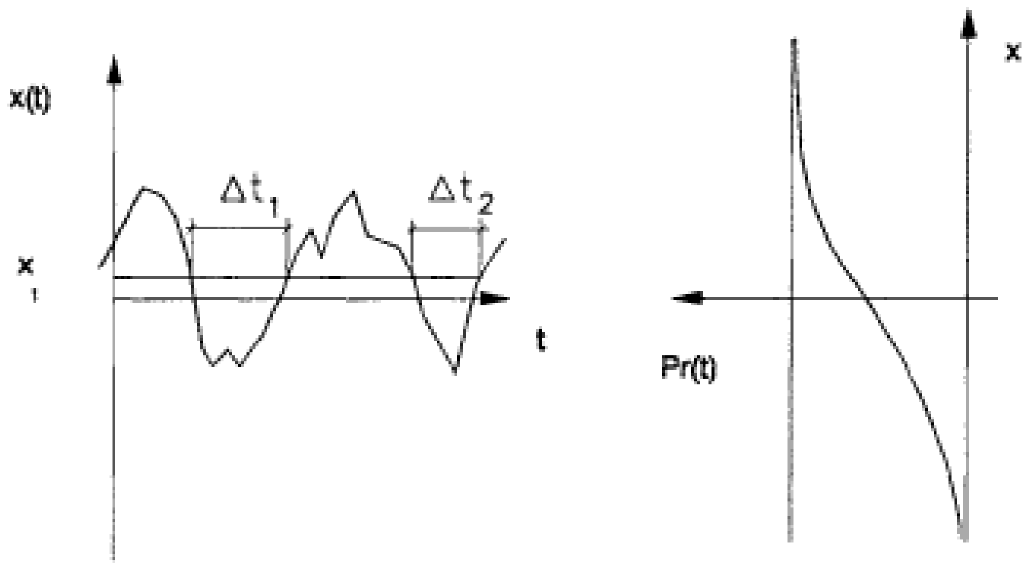
Probability Distribution and Density Functions:

Consider a sample of an ergodic process as shown in Fig. 3. We define the probability distribution function as

$$P(x) = \text{Prob}[x(t) < x] = \lim_{T\to\infty} \frac{1}{T} \sum_i \Delta t_i$$

We will define the probability density function as

$$p(x) = \lim_{\Delta x\to 0} \frac{P(x+\Delta x) - P(x)}{\Delta x} = \frac{\mathrm{d}P(x)}{\mathrm{d}x}$$



3 Probability distribution function

We verify the following relations:

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} p(x)\,\mathrm{d}x$$

$$p(-\infty) = p(\infty) = 0$$

$$P(x) \geq 0$$

$$P(x) = \int_{-\infty}^{x} p(\xi)\,\mathrm{d}\xi$$

$$P(\infty) = \int_{-\infty}^{\infty} p(\xi)\,\mathrm{d}\xi = 1$$

In many statistical applications where the number of samples is very great and none of the samples represents a significant weight in the process, the probability density function can be represented by the so-called Gaussian distribution. The probability density function for the Gaussian or normal distribution reads

$$p(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{(x-\underline{x})^2}{2\sigma_x^2}}$$

And thus the probability distribution function is given by

$$P(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-\underline{x})^2}{2\sigma_x^2}} \,\mathrm{d}x$$

Description of the Mean Values in Terms of the Probability Density Function:

Considering a stationary random process *(x(t)}* for a continuous function *g(x)*, we can write the mean value *g(x)* as

$$\underline{g(x)} = \frac{1}{n}\sum_{1}^{n}{}_{n\to\infty} g(x) = \sum_{1}^{n}{}_{n\to\infty} g(x)\frac{1}{n}$$

We note that *{ l / n }* represents the probability of the process to have the value of g(x). Thus, we can write

$$\underline{g(x)} = \sum_{-\infty}^{\infty}{}_{\Delta x\to 0} g(x)p(x)\Delta x = \int_{-\infty}^{\infty} g(x)p(x)\,\mathrm{d}x$$

We call *g(x)* the mean value or the mathematical expectation, and we write

$$\underline{g(x)} = E[g(x)]$$

Thus, we can write for the mean values the following expressions in terms of the probability density function:

1) For the mean value *g(x) = x,*

$$E[x] = \underline{x} = \int_{-\infty}^{\infty} x p(x)\,\mathrm{d}x$$

2) For the mean square value *g(x) = x2,*

$$E[x^2] = \underline{x^2} = \int_{-\infty}^{\infty} x^2 p(x)\,\mathrm{d}x$$

3) For the variance *g(x) = (x — x)2,*

$$\sigma_x^2 = E[(x-\underline{x})^2] = \int_{-\infty}^{\infty} (x-\underline{x})^2 p(x)\,\mathrm{d}x$$

$$\underline{x^2} - (\underline{x})^2 = E[x^2] - (E[x])^2$$

Properties of the Autocorrelation Function:

The autocorrelation function for an ergodic process reads

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\,dt$$

Making the transformation *t* — *r* = X, we get

$$R_x(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2-\tau}^{T/2-\tau} x(\lambda+\tau)x(\lambda)\,d\lambda$$

and because the integration is made for *T* -> ∞, we can write

$$R_x(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(\lambda+\tau)x(\lambda)\,d\lambda$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau)x(t)\,dt$$

Hence we conclude that the autocorrelation function is an even function.

Power Spectral Density Function:

Consider the sample *f (t)* of an ergodic process and its autocorrelation function, which can be written as

$$R_f(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)f(t+\tau)\,dt$$

This implies that the autocorrelation function is the inverse Fourier transform of *Sf(ω),* or

$$R_f(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_f(\omega)e^{i\omega\tau}\,d\omega$$

and we observe the following:

Sf(a>) does not furnish any new information since *Rf(ω)* is its Fourier transform, and thus the information contained in one is the same as the information contained in its transform. However, *Sf(co)* gives us the information in the frequency domain while *Rf(r)* gives us the information in the time domain, and depending on the application, one may be more convenient than the other.

Properties of the Power Spectral Density Function:

1 The Power Spectral Density Function Is a Positive Function

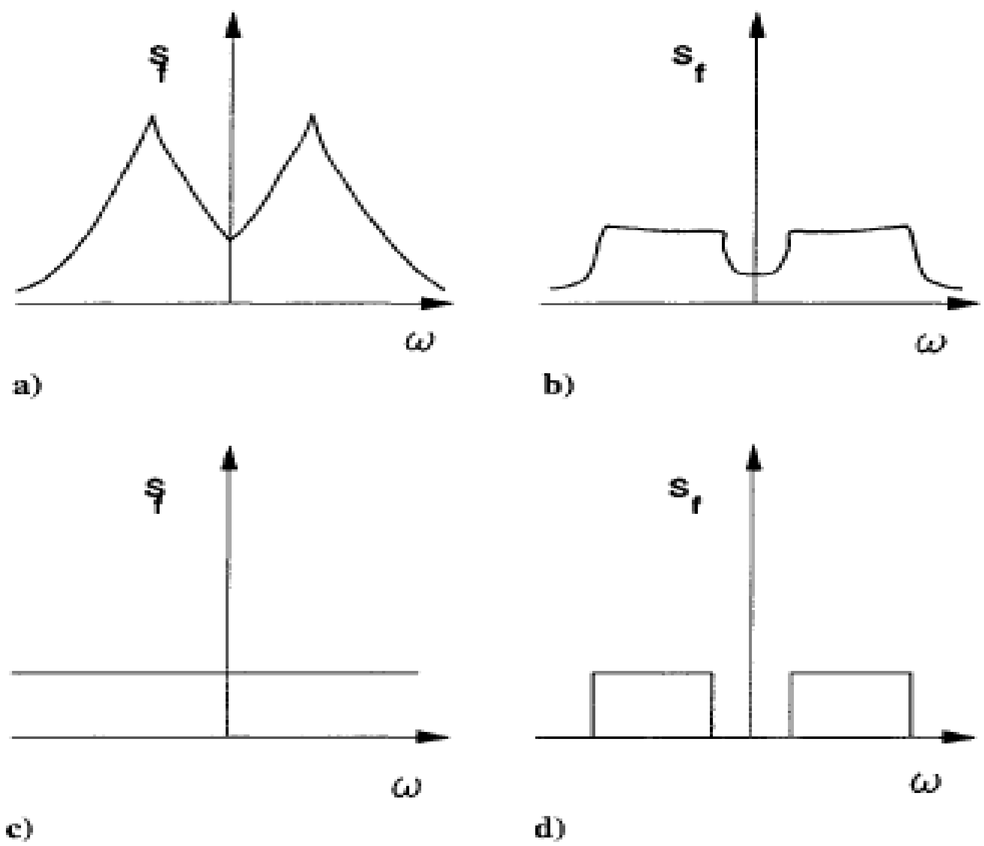
2 The Power Spectral Density Function Is an Even Function

3 Representation of the Power Spectral Density Function in the Positive Domain

White Noise and Narrow and Large Bandwidth:

The power spectral density function provides the necessary information on the frequency decomposition of a random process. Now if the frequency decomposition is concentrated in turns of a peak frequency a)Q as shown in Fig.a, we call such distribution a narrow bandwidth distribution. This is in contrast to the distribution given in Fig.4b, where we have an equal frequency distribution in a large band, and we call such distribution a large bandwidth distribution. Now, if Sf(a>) is a constant for all the frequency decompositions, i.e., from -∞ to ∞ as shown in Fig.4c,

We define such distribution as white noise; this is in comparison with the white light distribution, which has a plain spectral distribution in the large visible band frequency. In many practical cases, processes having distributions as shown in Fig.4d with an equal distribution in a large band of frequency can be considered as white noise distribution for practical purposes.



Narrow, large bandwidth and white noise distributions

Single-Degree-of-Freedom Response:

The response *x(t)* of a linear single-degree-of-freedom system due to an external applied load *f(t)*, whether a deterministic or random excitation, can be written in terms of Duhamel's convolution integral as

$$x(t) = \int_0^t f(t)h(t-\tau)\,d\tau = \int_0^t f(t-\lambda)h(\lambda)\,d\lambda$$

Now, for random excitation, we can extend the integration to —∞, and we write

$$x(t) = \int_{-\infty}^t f(t-\lambda)h(\lambda)\,d\lambda$$

The Fourier transform of the response reads

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt$$

Considering now a random ergodic excitation *f (t)* to a single-degree-of- freedom mechanical system, we can write the mean value of the response jc as

$$\underline{x} = E[x(t)] = E\int_{-\infty}^{\infty} h(\lambda)f(t-\lambda)\,d\lambda$$

And, because the system is linear, we can invert the order of the mean and the integration operations to write

$$\underline{x} = E[x(t)] = \int_{-\infty}^{\infty} E[h(\lambda)f(t-\lambda)\,d\lambda]$$

In the sequel, we will calculate the autocorrelation function of the response to a single degree of freedom due to an ergodic external excitation. Using Eq. we can write

$$x(t) = \int_{-\infty}^{\infty} f(\lambda_1)h(t-\lambda_1)\,d\lambda_1$$

$$x(t+\tau) = \int_{-\infty}^{\infty} f(\lambda_2)h(t+\tau-\lambda_2)\,d\lambda_2$$

Using the definition of the power spectral density function and Eq. we can write

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-i\omega\tau}\,d\tau$$
$$= \iint_{-\infty}^{\infty} e^{-i\omega\tau}[h(\lambda_1)h(\lambda_2)R_f(\tau+\lambda_1-\lambda_2)\,d\lambda_1\,d\lambda_2]\,d\tau$$

We conclude that

$$S_x(\omega) = |H(\omega)|^2 S_f(\omega)$$

It represents an algebraic relation between three functions, is a very important relation in structural dynamics.

Response to a White Noise:

Consider a single-degree-of-freedom mechanical system subjected to an external random ergodic excitation having a power spectral density function given by a white noise with intensity so- Thus, we can write

$$S_f(\omega) = S_0$$

Now, for a single-degree-of-freedom system, the complex frequency response function H(ω) reads

$$H(\omega) = \frac{1/k}{(1-\Omega^2) + 2i\gamma\Omega}$$

The autocorrelation function of the response can be obtained from the inverse Fourier transform of *Sx(a>)* and reads

$$R_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1/k)^2 S_0 e^{i\omega\tau}\,\mathrm{d}\omega}{[(1-\Omega^2)^2 + 4\gamma^2\Omega^2]}$$
$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{(1/k)^2 S_0 e^{i\omega\tau}\,\mathrm{d}\omega}{[(1-\Omega^2)^2 + 4\gamma^2\Omega^2]} \qquad \text{for } \tau \geq 0$$

Integrating, we obtain

$$R_x(\tau) = \frac{S_0\omega_n e^{-\gamma\omega_n\tau}}{4\gamma k^2}\left[\cos\omega_d\tau + \frac{\gamma}{(1-\gamma^2)^{\frac{1}{2}}}\sin\omega_d\tau\right] \qquad \text{for } \tau \geq 0$$

And the mean square value of the response reads

$$\psi_x^2 = R_x(0) = \frac{S_0\omega_n}{4\gamma k^2}$$

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